Hardness Beyond 2×2 Games

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Hardness beyond 2×2 games

Algorithms

Two example algorithms for finding NE

- LCP (Linear Complementarity) formulation
  - [Lemke-Howson '64]
- Support Enumeration Method
  - [Porter et al. '04]

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Early History (skipped in Basic lecture)

- **1928 von Neumann**: existence of Equilibrium in 2-player, zero-sum games
  - proof uses Brouwer's fixed point theorem;
  - led directly to algorithms:
    - Danzig '57: equivalent to LP duality
    - Khachiyan'79: polynomial-time solvable

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    - Danzig '57: equivalent to LP duality
    - Khachiyan'79: polynomial-time solvable
- **1950 Nash**: existence of Equilibrium in multiplayer, general-sum games
  - proof also uses Brouwer's fixed point theorem;
  - intense effort on equilibrium algorithms:
    - Kuhn '61, Mangasarian '64, Lemke-Howson '64, Rosenmüller '71, Wilson '71, Scarf '67, Eaves '72, Laan-Talman '79, Porter et al. '04, …
  - … all exponential in the worst case

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The Lemke-Howson Algorithm (skipped in Basic lecture)

- LCP (Linear Complementarity) formulation

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^* \qquad \forall j \in A_1$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k) \cdot s_1^j + r_2^k = U_2^* \qquad \forall k \in A_2$$

$$\sum_{j \in A_1} s_1^j = 1, \quad \sum_{k \in A_2} s_2^k = 1$$

$$s_1^j \geq 0, \;\; s_2^k \geq 0 \qquad \forall j \in A_1, \; \forall k \in A_2$$

$$r_1^j \geq 0, \;\; r_2^k \geq 0 \qquad \forall j \in A_1, \; \forall k \in A_2$$

$$r_1^j \cdot s_1^j = 0, \;\; r_2^j \cdot s_2^j = 0 \qquad \forall j \in A_1, \; \forall k \in A_2$$

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Support Enumeration Method: Porter *et al.* 2004 (skipped in Basic lecture)

- Step 1: Finding a NE with a specific support

$$\sum_{a_{-1} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) = v_i \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$\sum_{a_{-1} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) \leq v_i \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$p_i(a_i) \geq 0 \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$p_i(a_i) = 0 \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$\sum_{a_i \in \sigma_i} p_i(a_i) = 1 \qquad \forall i \in \{1,2\}$$

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Support Enumeration Method: Porter *et al.* 2004 (skipped in Basic lecture)

- Step 2: Smart heuristic search through all sets of support

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From Algorithms to Complexity Analysis (skipped in Basic lecture)

- These algorithms have exponential worst-case time complexity.
- So do all known others.
- Can we do better?

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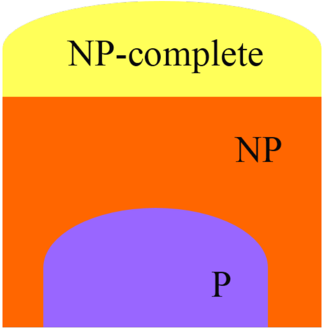
From Algorithms to Complexity Analysis (skipped in Basic lecture)

- Reminder of a (small part) of the complexity hierarchy.



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From Algorithms to Complexity Analysis (skipped in Basic lecture)

So, is it NP-complete to find a Nash equilibrium?

- Strictly speaking, no, since a solution is guaranteed to exist…
- However, it is NP-complete to find a "tiny" bit more info than a Nash equilibrium; e.g., the following are NP-complete:
  1. **(Uniqueness)** Given a game G, does there exist a unique equilibrium in G?
  2. **(Pareto optimality)** Given a game G, does there exist a strictly Pareto efficient equilibrium in G?
  3. **(Guaranteed payoff)** Given a game G and a value v, does there exist an equilibrium in G in which some player i obtains an expected payoff of at least v?
  4. **(Guaranteed social welfare)** Given a game G, does there exist an equilibrium in which the sum of agents' utilities is at least k?
  5. **(Action inclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in N$, does there exist an equilibrium of G in which player i plays action a_i with strictly positive probability?
  6. **(Action exclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in N$, does there exist an equilibrium of G in which player i plays action a_i with zero probability?

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From Algorithms to Complexity Analysis

Still, finding even a single Nash equilibrium seems hard; how do we capture that?

- Enter PPAD (“Polynomial Parity Arguments on Directed graphs”)
  - (Papadimitriou ‘94)
- At a high level:
  - FNP problems are constructive versions of NP problems (F stands for “Functional”)
  - TFNP is a subclass of FNP for problems for which a solution is guaranteed to exist (T stands for “Total”)
  - PPAD is a subclass of TFNP where the proofs are based on parity arguments in directed graphs

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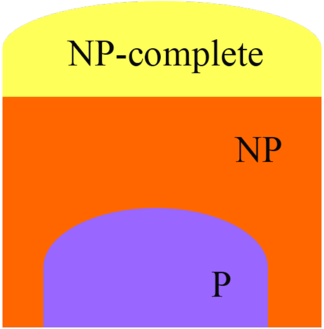
From Algorithms to Complexity Analysis

Where is PPAD?



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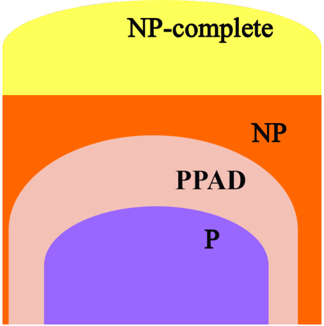

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From Algorithms to Complexity Analysis

The Complexity of the Nash Equilibrium

Theorem: Computing a Nash equilibrium is PPAD-complete…

- for games with ≥4 players;
  [Daskalakis, Goldberg, Papadimitriou '05]
- for games with 3 players; [Chen, Deng '05] &
  [Daskalakis, Papadimitriou '05]
- for games with 2 players. [Chen, Deng '06]